NC STATE UNIVERSITY

College of Engineering

Department of Mechanical and Aerospace Engineering

MAE-315, Section 001

Dynamics of Machines

Dr. Fen Wu

Course Project

Written By: **David Delgado**
Marelis Perez-Hernandez
Justin Powers

Date Report Submitted: **12/05/2018**

Abstract

The objective of this experiment was to become familiar with analyzing and visualizing a multiple degree of freedom vibrational system. In addition to learning how to apply different analysis techniques (i.e., Lagrange's equation, Newton's Law and MATLAB) in solving the system, a design suggestion to reduce the maximum machine tool head vibration by 25% was formulated through an analysis. To meet the design criteria, changing the value of stiffness k_3 from 2000 lb/in to 400 lb/in was found to have the best effect with little system repercussions, and required the least change in the magnitude of the initial value (133.33% difference). The percent differences for different design changes can be seen in Table 1. Meanwhile, adjusting stiffness k_2 was found to be the least favorable method as it would require the most change (193.02% difference) and caused additional vibration throughout the system.

1. Introduction

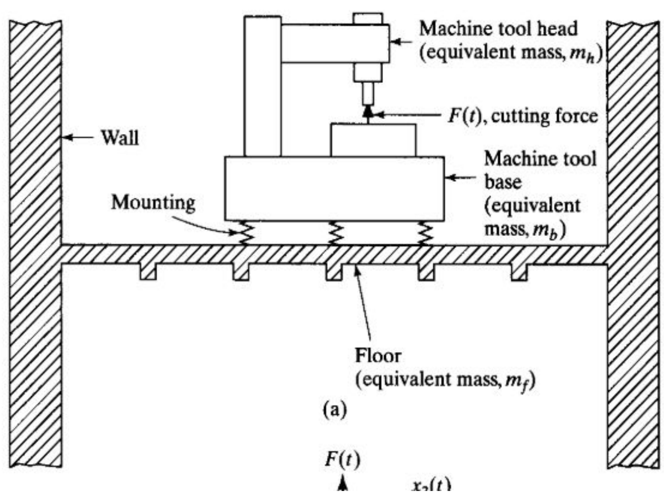

Figure 1.1

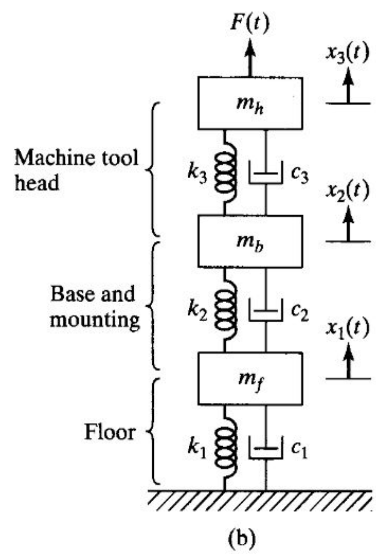

Figure 1.2

The purpose of the project was to determine the vibrational response of the 3-DOF system displayed in Figure 1.1. Figure 1.2 is the simplified diagram, which was used for EOM derivations and steady-state vibration calculations. Additionally, the project advanced an understanding of the relationship between the spring constant and vibrational response, as well as the damping coefficient and vibrational response, as different ways of reducing the maximum response of the machine tool head (x_3) were compared.

2. Experimental Methods

The analysis of this system began with deriving the equations of motion (EOM) using Lagrange's equation, and using those EOMs to solve for the steady-state response of the system through the mechanical impedance method and Cramer's Rule. The EOMs were verified using Newton's Second Law. Then the steady-state responses were solved and plotted in MATLAB, and the solutions were compared to ensure accuracy. Using code to iteratively solve for the maximum response of the tool head, a value of k_2 that would ensure a 25% reduction in amplitude of the tool head was determined. Lastly, other methods of reducing the amplitude of vibration of the tool head were investigated and compared in Table 1.

3. Experimental Data

The derivation of the EOMs was conducted using kinetic energy equations to describe masses, potential energy equations to describe springs, and the Rayleigh dissipation function to describe dampers, and then substituting those equations into Lagrange's Equation, given below:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \left(\frac{\partial T}{\partial x_i}\right) + \left(\frac{\partial R}{\partial \dot{x}_i}\right) + \left(\frac{\partial V}{\partial x_i}\right) = F_i$$

Where T is the equation for kinetic energy, R is Rayleigh's dissipation function, and V is the potential energy function.

The EOM was determined to be, in matrix form:

$$\begin{bmatrix} m_f & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & m_h \end{bmatrix}\begin{bmatrix} \ddot{x_1} \\ \ddot{x_2} \\ \ddot{x_3} \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f_3(t) \end{bmatrix}$$

Next, the steady-state vibration of the system was calculated by defining the mechanical impedance matrix as:

$$[Z(i\omega)] = \begin{bmatrix} -\omega^2 m_f + i\,\omega\,(c_1 + c_2) + (k_1 + k_2) & -i\,\omega\,c_2 - k_2 & 0 \\ -i\,\omega\,c_2 - k_2 & -\omega^2 m_b + i\,\omega\,(c_2 + c_3) + (k_2 + k_3) & -i\,\omega\,c_3 - k_3 \\ 0 & -i\,\omega\,c_3 - k_3 & -\omega^2 m_h + i\,\omega\,c_3 + k_3 \end{bmatrix}$$

The EOM was rewritten as:

$$[Z(iw)]\vec{X} = \vec{F}_0$$

Once the equation above was solved for values of X_j utilizing Cramer's rule, the steady-state solution was calculated by using the following:

$$x_j(t) = X_j(\cos(60t) + i\sin(60t)), \qquad j = 1, 2, 3$$

Keeping only the real numbers to describe the actual vibration of the system, and rewriting the equations in terms of a magnitude and a phase angle, the steady-state vibration was determined to be:

$$x_1(t) = (-1.2298\cos(60t) + 5.2840\sin(60t)) * 10^{-5}\ in = (5.4252\cos(\omega t + 76.898°)) * 10^{-5}\ in$$

$$x_2(t) = (0.0109\ \cos(60t) - 0.0054\sin(60t))\ in = (0.012164\cos(\omega t + 26.354°))\ in$$

$$x_3(t) = (-0.1929\cos(60t) + 0.0256\sin(60t))\ in = (0.19459\cos(\omega t + 7.5596°))\ in$$

The following code was used to solve and verify the steady-state motions of the system. Plots of the individual and system responses were also generated to visualize the motion.

```
% Define known variables
mf = 50;
mb = 10;
mh = 2;

%units are lb*sec^2/in

c1 = 10;
c2 = 10;
c3 = 10;

%units are lb*sec/in

k1 = 5000;
k2 = 500;
k3 = 2000;

%units are lb/in

w = 60;

%units are in rad/sec

%Create matrices for constants and forces
M = [mf, 0, 0; 0, mb, 0; 0, 0, mh];
C = [c1+c2, -c2, 0; -c2, c2+c3, -c3; 0, -c3, c3];
K = [k1+k2, -k2, 0; -k2, k2+k3, -k3; 0, -k3, k3];

F = [0;0;1000];

%Impedance Matrix
for i = 1:3
    for J = 1:3
        Z(i,J) = -w^2*M(i,J) + j*w*C(i,J) + K(i,J);
    end
end

%Phase Angle
X = inv(Z)*F;
magX = abs(X);
theta = angle(X)*(180/pi);


%Time Interval
int = .1;
t = 0:int:30;

%Steady State Solutions
X1 = magX(1)*cos(t+angle(1));
X2 = magX(2)*cos(t+angle(2));
X3 = magX(3)*cos(t+angle(3));
```

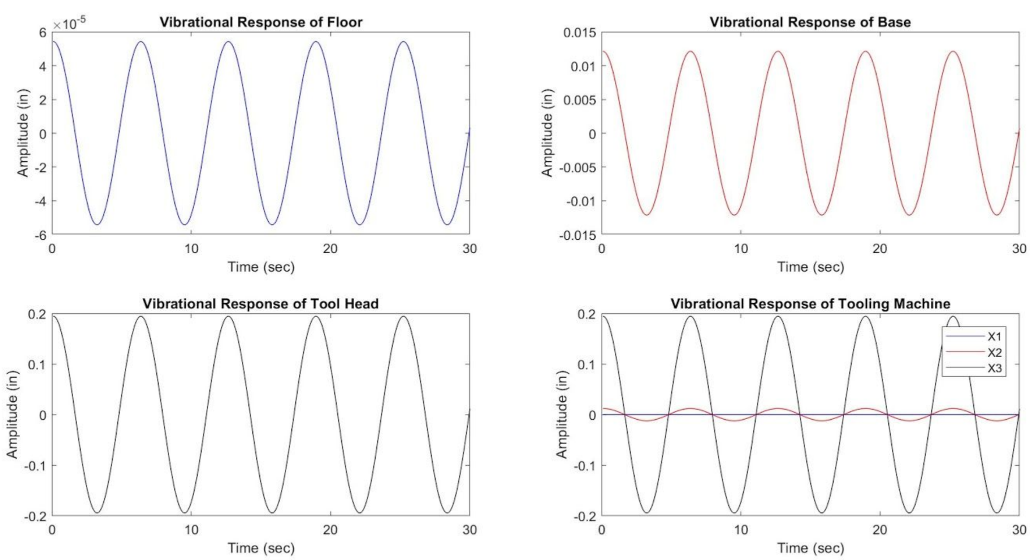

Figure 3.1. Plots of the Steady-State Response of System

4. Theory and Analysis

After solving the system, different values of k_2 that would decrease the amplitude of the machine tool head (x_3) by 25% (i.e., from a value of 0.1946 in. to 0.1459 in.) were tested. The first idea was to initiate a trial-and-error process, which was a failure in that the new values of k_2 guessed (1000 lb/in, 150,000 lb/in, 1,000,000 lb/in) all returned values above 0.190 in. for x_3. The second approach was to generate two matrices in MATLAB --one of potential k_2 values from 0 to 1,000,000 lb/in in increments of 100 lb/in and one of the corresponding x_3 values-- using the following additions and edits to the existing code.

```
k1 = 5000;
k22 = [0:100:1000000];
k3 = 2000;

%units are
testmag = ones(size(k22));
[row, col] = size(k22);

for d = 1:col
    k2 = k22(d);
```

```
magX(3)
testmag(d) = magX(3);
end
```

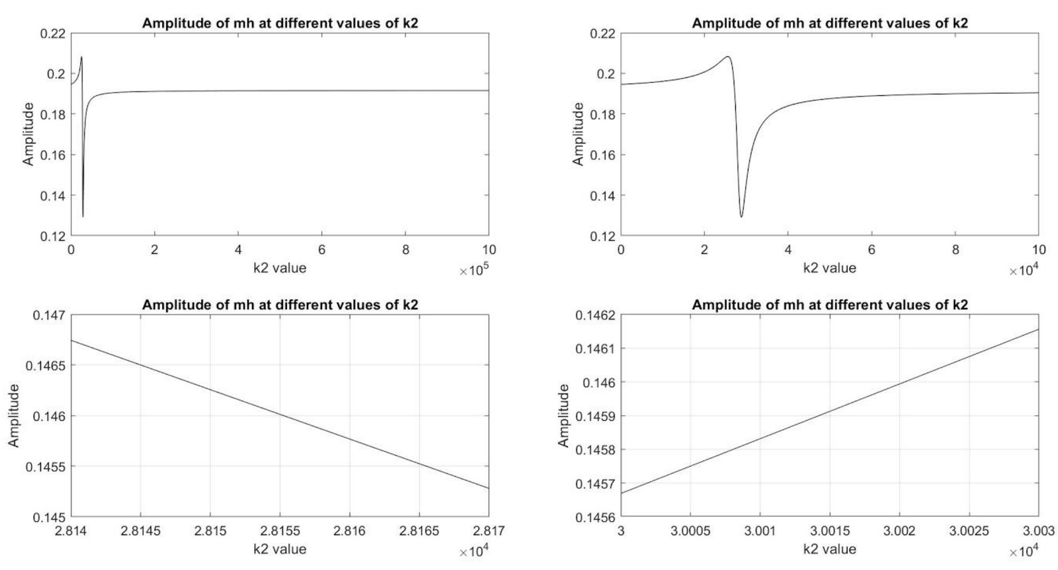

Figure 4.1. Maximum Responses of the Tool Head at Different k_2 Values

Then the relationship between k_2 and x_3 was plotted, as seen in Figure 4.1. The pattern was remarkable. The amplitude of x_3 drastically increased, then went through a valley over an intermediate set of values with a minimum at 0.1291, and then for extremely large values leveled off asymptotically around 0.194. The two values of k_2 that provided an acceptable x_3 ($\approx$.0146) are displayed in the bottom two quadrants of the figure. The hypothesis is that for those extremely large spring constants, the spring stops acting as a shock absorption device and begins to act like a rigid body, which offers little spring action, and explains the asymptotic behavior towards the right side of the graph. Overall, it was found that reducing the amplitude of x_3 could be achieved by increasing the spring constant k_2 to either 28,155 lb/in or 30,020 lb/in. Similarly, other ways of reducing x_3 were investigated, and documented in Table 1.

Table 1. Comparison of Isolated Vibration Reduction Methods

Variable Manipulated	Initial Value	New Value[1]	% difference[2]
k_1	5000 lb/in	Not attainable[3]	N/A
k_2	500 lb/in	28155 lb/in, 30020 lb/in	193.02%, 193.45%
k_3	2000 lb/in	400 lb/in, 10030 lb/in	133.33%, 133.50%
c_1	10 lb-sec/in	Not attainable	N/A
c_2	10 lb-sec/in	Not attainable	N/A
c_3	10 lb-sec/in	61 lb-sec/in	143.66%

Footnotes:

1. The approximate required values in order to reduce the maximum response of x_3 by 25% (i.e., x_3 = 0.14594 in.)
2. Percent difference between the new and initial values as a result of achieving the desired vibration reduction is calculated using:

$$\% \, diff = \frac{|Value_{new} - Value_{initial}|}{\frac{1}{2}(Value_{new} + Value_{initial})} * 100\%$$

3. It is not possible to achieve a 25% reduction in the response of x_3 given the isolated manipulation of this variable.

From the results of the isolated vibration reduction trials it was determined that the three different ways to achieve a 25% reduction in the vibration response x_3 were by individually manipulating k_2, k_3, and c_3. The effect of changing the values of these variables in order to achieve this goal on the vibration response of the entire system is shown in Figure 4.2. Clearly, changing the stiffness k_2 to the required value does not have a positive effect on the response of the machine tool base and floor. This method would not be favorable as it also entails the largest percent difference values, meaning that it involves the most drastic changes in the magnitude of the spring stiffness. The second method investigated was changing k_3, which appears to reduce the maximum response of not only the machine tool head but also the base. Hence, this method is an improvement from the base case and corresponds to a less drastic change in the magnitude of the spring stiffness (i.e., 133.33% difference). Lastly, the damping coefficient c_3 was changed and it appeared to change only the maximum response of the machine tool head, which is desirable if the initial response of the other two parts of the machine is expected. Nevertheless, this method involves a larger change in the magnitude of the damping coefficient (i.e., 143.66% difference).

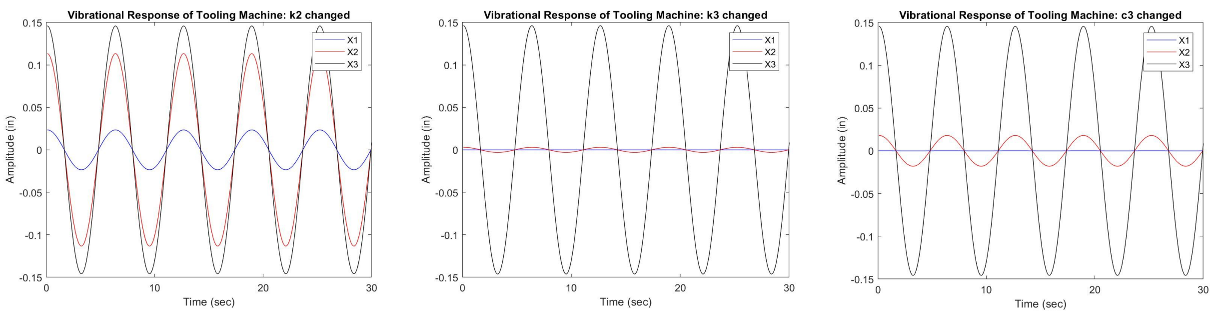

Figure 4.2. Responses Under Various Changes to System Constants

5. Conclusions

From the results of the steady-state solutions, it is obvious that the tooling head experiences the most motion, and then the base, and then the floor, due to the force acting directly on the tooling head. All of the amplitudes were less than two tenths of an inch, which indicates that the machine is not shaking on the floor, but the response of the entire system may want to be kept in mind when thinking of tolerances. Hence, when looking to reduce the vibration of the tooling head other options should be considered beyond altering k_2, since it has feedback effects on the machine tool base and causes it to vibrate at a higher amplitude. The better method would be to adjust k_3 or c_3, which are more directly connected to the tooling head and therefore require a smaller change in the magnitude of their values to achieve a 25% reduction in the maximum response thereof.

The best method would be to adjust k_3 from 2000 lb/in to 400 lb/in, as it requires the least change in magnitude, it has a damping effect on the other system components, and would be the easiest to manipulate in a real manufacturing setting. Still, this recommendation is based on an oversimplification of the actual mechanical system, which assumes a linear relationship in the effects of the spring and dampers and that it can be modeled as a parallel circuit. For further study, the analysis of non-isolated vibration reduction could be included in the investigation (i.e., manipulating k_3 and c_3 simultaneously) and an analysis of these changes on the natural frequency of vibration of the system would also be useful.

6. References

[1] Rao, S.S., (2016). *Mechanical Vibrations* (6th ed.). Pearson-Prentice Hall.

[2] Wu, Fen (2018). *MAE 315-1 Dynamics of Machines Vibrations.* Department of Mechanical and Aerospace Engineering. North Carolina State University. Lecture Notes 1-28.

[3] Thorby, D. (2008). 11. *Vibration Reduction. Structural Dynamics and Vibration in Practice - an Engineering Handbook () Elsevier.*

7. Sample Calculations

[Primary Contributor: Justin Powers]

These calculations expand on the derivation of the EOM in the Experimental Data section:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \left(\frac{\partial T}{\partial x_i}\right) + \left(\frac{\partial R}{\partial \dot{x}_i}\right) + \left(\frac{\partial V}{\partial x_i}\right) = F_i$$

Where T is the equation for kinetic energy, R is Rayleigh's dissipation function, and V is the potential energy function.

Kinetic Energy Component

Kinetic Energy Equation

$$T = \tfrac{1}{2}m_f\dot{x}_1^2 + \tfrac{1}{2}m_b\dot{x}_2^2 + \tfrac{1}{2}m_h\dot{x}_3^2$$

Matrix Form

$$T = \frac{1}{2}\begin{bmatrix}\dot{x}_1 & \dot{x}_2 & \dot{x}_3\end{bmatrix}\begin{bmatrix}m_f & 0 & 0\\ 0 & m_b & 0\\ 0 & 0 & m_h\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}$$

From lecture slides 19-21, slide 8[2]:

$\frac{\partial T}{\partial \dot{x}_i} = \frac{1}{2}\delta_i^T[M]\dot{x} + \frac{1}{2}\dot{x}^T[M]\delta_i = M_i\dot{x}$; differentiating and concatenating, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = [M][\ddot{x}]$

$\frac{\partial T}{\partial x_i} = 0$, as T has no dependence on x, only $\dot{x}$.

Potential Energy Component

Potential Energy Equation

$$V = \tfrac{1}{2}k_1x_1^2 + \tfrac{1}{2}k_2(x_2 - x_1)^2 + \tfrac{1}{2}k_3(x_3 - x_2)^2$$

Matrix Form

$$V = \frac{1}{2}\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}k_1+k_2 & -k_2 & 0\\ -k_2 & k_2+k_3 & -k_3\\ 0 & -k_3 & k_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

From lecture slides 19-21, slide 8[2]:

$$\frac{\partial V}{\partial x_i} = K_i x \text{ ; concatenating, } \frac{\partial V}{\partial x} = [K][x]$$

Rayleigh's Dissipation Function[1]

Rayleigh Function

$$R = \tfrac{1}{2}c_1\dot{x}_1^2 + \tfrac{1}{2}c_2(\dot{x}_2 - \dot{x}_1)^2 + \tfrac{1}{2}c_3(\dot{x}_3 - \dot{x}_2)^2$$

Matrix Form

$$R = \frac{1}{2}\begin{bmatrix}\dot{x}_1 & \dot{x}_2 & \dot{x}_3\end{bmatrix}\begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3\end{bmatrix}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3\end{bmatrix}$$

From the kinetic energy equation calculations above:

$$\frac{\partial R}{\partial \dot{x}_i} = \tfrac{1}{2}\delta_i^T[C]\dot{x} + \tfrac{1}{2}\dot{x}^T[C]\delta_i = C_i\dot{x} \text{ ; concatenating, } \left(\frac{\partial R}{\partial \dot{x}}\right) = [C][\dot{x}]$$

To derive EOM, substitute the concatenated partial derivatives into Lagrange's Equation.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \left(\frac{\partial T}{\partial x_i}\right) + \left(\frac{\partial R}{\partial \dot{x}_i}\right) + \left(\frac{\partial V}{\partial x_i}\right) = F_i \;\rightarrow\; [M][\ddot{x}] - 0 + [C][\dot{x}] + [K][x] = [F]$$

$$\begin{bmatrix} m_f & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & m_h\end{bmatrix}\begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3\end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3\end{bmatrix}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3\end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} = \begin{bmatrix}0 \\ 0 \\ f_3(t)\end{bmatrix}$$

From Section 5.6 of the Mechanical Vibrations textbook, the forcing function can be written as:

$$f_3(t) = F_{30}e^{i\omega t}$$

Furthermore, the steady-state solution can be assumed as:

$$f_3(t) = F_{30}e^{i\omega t}$$

Thus, the EOM becomes

$$\begin{bmatrix} -\omega^2 m_f + i\,\omega\,(c_1 + c_2) + (k_1 + k_2) & -i\,\omega\,c_2 - k_2 & 0 \\ -i\,\omega\,c_2 - k_2 & -\omega^2 m_b + i\,\omega\,(c_2 + c_3) + (k_2 + k_3) & -i\,\omega\,c_3 - k_3 \\ 0 & -i\,\omega\,c_3 - k_3 & -\omega^2 m_h + i\,\omega\,c_3 + k_3\end{bmatrix}\begin{bmatrix}X_1 \\ X_2 \\ X_3\end{bmatrix} = \begin{bmatrix}0 \\ 0 \\ F_{30}\end{bmatrix}$$

Given the definition of mechanical impedance, the previous equation becomes

$$[Z(iw)]\vec{X} = \vec{F}_0$$

where

$Z_{11} = -174{,}500 + i\ 1{,}200$

$Z_{12} = -500 - i\ 600$

$Z_{13} = Z_{31} = 0$

$Z_{22} = -33{,}500 + i\ 1{,}200$

$Z_{23} = Z_{32} = -2{,}000 - i\ 600$

$Z_{33} = -5{,}200 + i\ 600$

$F_{30} = 1000$

Hence, using Cramer's rule, the solution of the vector X can be obtained as follows:

$$X_1 = \begin{bmatrix} 0 & Z_{12} & Z_{13} \\ 0 & Z_{22} & Z_{23} \\ F_{30} & Z_{32} & Z_{33} \end{bmatrix} \frac{1}{\det[Z(iw)]} = (-1.2298 - 5.2840\ i) * 10^{-5}$$

$$X_2 = \begin{bmatrix} Z_{11} & 0 & Z_{13} \\ Z_{21} & 0 & Z_{23} \\ Z_{31} & F_{30} & Z_{33} \end{bmatrix} \frac{1}{\det[Z(iw)]} = (0.0109 - 0.0054\ i)$$

$$X_3 = \begin{bmatrix} Z_{11} & Z_{12} & 0 \\ Z_{21} & Z_{22} & 0 \\ Z_{31} & Z_{32} & F_{30} \end{bmatrix} \frac{1}{\det[Z(iw)]} = -0.1929 - 0.0256\ i$$

This result is used to calculate the steady-state solution using the following equation:

$$x_j(t) = Real\ [X_j(\cos(60t) + i\sin(60t))], \qquad j = 1, 2, 3$$

In this way, the following actual steady-state responses are calculated:

$$x_1(t) = \left(-1.2298\ cos\ (60t) + \ 5.2840\ sin(60t)\right) * 10^{-5}\ in$$

$$x_2(t) = \left(0.0109 cos\ (60t) - \ 0.0054\ sin(60t)\right)\ in$$

$$x_3(t) = \left(-0.1929\ cos\ (60t) + \ 0.0256\ sin(60t)\right)\ in$$

From lecture slides 3-4, the alternative solution form of the EOM leads to the following responses:

$$x_1(t) = (5.4252\cos(\omega t + 76.898°)) * 10^{-5}\ in$$

$$x_2(t) = (0.012164\cos(\omega t + 26.354°))\ in$$

$$x_3(t) = (0.19459\cos(\omega t + 7.5596°))\ in$$

Lastly, the percent difference between new and initial values of variables manipulated to achieve a 25% reduction in the vibration response of the machine tool head was calculated as such:

$$\%\ diff = \frac{|Value_{new} - Value_{initial}|}{\frac{1}{2}(Value_{new} + Value_{initial})} * 100\%$$